Basic example & existence of non-measureable sets

math center

We will consider the Borel σ-algebra on $\mathbb{R}$ and its completion. The measure is always denoted by m. We begin by reviewing the easiest example of a non-measurable set, the Vitali set.

The idea is easy to follow: we find a countable partition of $[0,1]$ into closely related sets. Choose a subset $A \subset [0,1]$ which contains exactly one element in each class of $\mathbb{R}/\mathbb{Q}$. Let $0 \in A$ for convenience. For rational numbers $q \in [0,1)$, define

$$A_q = [0,1] \cap ((q+A) \cup (q+A-1)).$$

Since $A \subset [0,1)$, $q+A$ and $q+A-1$ are always disjoint. Given that m satisfies translation invariance, we have $m(A_q) = m(A)$ for all q.

A is known as the Vitali set. We showed that

Proposition 0.1. The Vitali set is not Lebesgue (Borel) measurable.

Proof. The $\{A_q\}$ form a disjoint partion of the interval $[0,1)$, which has measure 1. $\square$

Our next goal is to answer the following question—at least partially. Given a subset $E \subset \mathbb{R}$, is there a non-measurable subset of E?

When E is measurable has positive measure, it is actually not more difficult.

Proposition 0.2. If E is Lebesgue measureable subset of $[0,1]$ and $m(E) > 0$, then $E \cap A$ is a non-measurable set.

Proof. The translations $E_q := q + E \cap A$, $q \in \mathbb{Q} \cap [0,1)$, are disjoint and their union is a set between E and $[0,2]$. Hence we get the same contradiction as above. $\square$

1 An existence result

If $\mathcal{P}(E)$ is too big in size, some subset of E can't be Borel measurable.

Proposition 1.1. Let S be an infinite subset of $\mathcal{P}(X)$, then the cardinality of the σ-algebra generated by S satisfies

$$|\mathscr{M}(S)| = |S| \cdot \aleph_1.$$

Proof. For all countable ordinals α, inductively define

- M_0 is the union of S with the set of subsets whose complement is in S;
- U_α denotes the set of countable unions of elements of M_α, and we put $M_{\alpha+1} := U_\alpha \cup \{B^c : B \in U_\alpha\}$;
- If $\alpha > 0$ is a limit ordinal, put $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

Every M_α is closed under taking complements, and has the same cardinality as $|S|$ if α is countable. By a cofinality argument $(\mathrm{cof}(\omega) > \omega)$, we see that $M_{\aleph_1}$ is an σ-algebra, hence it is $\mathscr{M}(S)$. $\square$

Corollary 1.2. The Borel σ-algebra $\mathcal{B}_\mathbb{R}$ has cardinality $\mathfrak{c} := |\mathbb{R}|$.

Proof. Yes. $\square$

A classical result from descriptive set theory states that any Borel set of a Polish space has the perfect set property; in particular, an uncountable Borel set in $\mathbb{R}^d$ must have cardinality $\mathfrak{c}$. Therefore it must contain a non-Borel-measurable set.

For other subsets of $\mathbb{R}^d$, as long as its power set has a cardinality larger than $\mathfrak{c}$, this will also hold.

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