Solving the homogeneous linear ODE in calculus, justified?

math center

Fix the field $k = \mathbb{R}$ or $\mathbb{C}$. We will be working with (vectors of) anayltic functions over k which we denote by $\mathcal{O}(-)$.

1 Homogeneous linear ODE

Theorem 1.1 (Cauchy)**.** Let $0 < R \leq \infty$ and $D := \{z \in k : |z| < R\}$. Suppose $A \in \mathrm{Mat}_{n\times n}(\mathcal{O}(D))$ is a matrix of analytic functions on D. Then given any $u_0 \in k^n$, there is a unique vector-valued analytic function $u \in \mathcal{O}(D)^n$ such that $u(0) = u_0$ and $u'(z) = A(z)u(z)$ for all $z \in D$.

Proof of Uniqueness. It is well known that any function in $\mathcal{O}(D)$ has a unique Taylor expansion centered at 0 that converges everywhere on D. Let's write

$$A(z) = \sum_{r=0}^{\infty} A_r z^r, \quad A_r = (a_{ij,r})_{ij} \in \mathrm{Mat}_{n\times n}(k),$$

and

$$u(z) = \sum_{r=0}^{\infty} u_r z^r, \quad u_r = (u_{i,r})_i \in k^n.$$

Note that this agrees with the given u_0. The equation $u'(z) = A(z)u(z)$ becomes

$$\sum_{r=0}^{\infty} r u_r z^{r-1} = \sum_{t=0}^{\infty} A_t \sum_{s=0}^{\infty} u_s z^{t+s} = \sum_{r=0}^{\infty}\left(\sum_{s=0}^{r} A_{r-s}u_s\right) z^r.$$

By comparing the coefficients of both sides, we deduce that

$$(r+1)u_{r+1} = \sum_{s=0}^{r} A_{r-s}u_s, \quad \forall r \geq 0. \tag{1}$$

This is a full recurrence relation for the u_r's. ☐

Proof of Existence. Let u_r be given by the recurrence relation (1) with the initial value u_0. We need to show that the series for $u(z)$ converges within D.

For any $0 < \rho < R$, the series

$$\sum_{r=0}^{\infty} |a_{ij,r}|\, \rho^r$$

converges for all i, j. Hence there exists a natural number N such that

$$|a_{ij,r}|\, \rho^r \leq N\rho^{-1}, \quad \forall i, j, r.$$

Define the following matrix

$$B(z) = \sum_{r=0}^{\infty} B_r z^r, \quad B_r = (b_{ij,r})_{ij},$$

where

$$b_{ij,r} := \frac{N}{\rho^{r+1}} \geq |a_{ij,r}|, \quad \forall i, j, r. \tag{2}$$

We then look for a solution of the form $v(z) = (f(z), \cdots, f(z))$ to the equation $v'(z) = B(z)v(z)$ within a smaller disk $D' = \{z : |z| < \rho\}$. As every entry of $B(z)$ is equal to

$$b(z) := \frac{N}{\rho}\left(1 - \frac{z}{\rho}\right)^{-1} \in \mathcal{O}(D'),$$

this is not difficult to solve:

$$\begin{gathered} f'(z) = nb(z)f(z) \\ \implies f(z) = C\exp\left\{n\int_0^z b(t)dt\right\} = C\left(1 - \frac{z}{\rho}\right)^{-nN} \in \mathcal{O}(D'). \end{gathered}$$

Here $C = f(0)$ is an initial value yet to be determined. Thus, applying (1) to this equation, we can write

$$v(z) = \sum_{r=0}^{\infty} v_r z^r, \quad (r+1)v_{r+1} = \sum_{s=0}^{r} B_{r-s}v_s, \quad \forall r \geq 0.$$

This completes the setup for Cauchy's majorization method. Once we set

$$C = v_{1,0} = \cdots = v_{n,0} \overset{!}{=} \max\{|u_{1,0}|, |u_{2,0}|, \cdots, |u_{n,0}|\} > 0,$$

we can show that every component of v_r is positive using induction. Then, by doing another induction on r with (2), we obtain

$$|u_{i,r}| \leq v_{i,r}, \quad \forall i, r.$$

Since $v(z) \in \mathcal{O}(D')^n$ converges, the series $u(z) = \sum_{r=0}^{\infty} u_r z^r$ converges on D' as well. This completes the proof. □

Corollary 1.2. The analytic solutions $y \in \mathcal{O}(D)$ to the following ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \quad a_i \in \mathcal{O}(D), \quad \forall i,$$

form a k-vector space of dimension n.

Proof. This equation is turned into the previous matrix equation by putting

$$u = \left(y, y', \cdots, y^{(n-1)}\right)^\top.$$

The matrix A is defined accordingly. The theorem gives an isomorphism between $\{u : u' = Au\}$ and k^n. □

2 A generalization

The theorem has a natural generalization to Riemann surfaces. Let M be a simply connected Riemann surface, $p \in M$.

Suppose $A \in \mathrm{Mat}_{n\times n}(\Omega^{1,0}(M))$ is a matrix of holomorphic 1-forms on M. We solve for a vector-valued holomorphic function $u \in \mathcal{O}(M)^n$ satisfying $u(p) = u_0$ and

$$du = Au. \tag{3}$$

For any open subset $U \subset M$, denote by $\mathcal{F}(U) \subset \mathcal{O}(U)$ the set of solutions to (3) on U. Then $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces on M.

Corollary 2.1. For each $u_0 \in k^n$, there exists a unique u satisfying the above conditions.

Proof. First, let's confirm that the equation locally reduces to the theorem. Around any point $m \in M$, we may take a holomorphic chart $z : U \xrightarrow{\sim} D$ where D is a disk in the complex plane centered at $z(m)$. Then matrix A can then be written as $A = \tilde{A}dz$ for some $\tilde{A} \in \mathrm{Mat}_{n\times n}(\mathcal{O}(U))$. Identifying U with D, the equation becomes

$$du = \tilde{A}udz \Longleftrightarrow u' = \tilde{A}u.$$

Hence the theorem applies, implying that $\mathrm{ev}_m : \mathcal{F}(U) \longrightarrow \mathbb{C}^n$ is an isomorphism of $\mathbb{C}$-vector spaces.

As a result, any two local solutions agreeing at a point m must coincide in a neighborhood of m. By basic complex analysis, they must coincide on the entire connected component containing m, wherever they are both defined. It follows that for any $m \in V$ with V open and connected, the evaluation map $\mathrm{ev}_m : \mathcal{F}(V) \longrightarrow \mathbb{C}^n$ is injective.

Moreover, consider open and connected subsets $\varnothing \neq V \subset U$, such that $\mathcal{F}(U) \simeq \mathbb{C}^n$. Then the injectivity implies that the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is a bijection. Since such U's cover M, we conclude that $\mathcal{F}$ is a locally constant sheaf.

Therefore the étale space of $\mathcal{F}$ is a covering space of M. Since M is simply connected, the covering space must be trivial. In other words, $\mathcal{F}$ is a constant sheaf. It follows that $\mathcal{F}(M) \simeq \mathbb{C}^n$. ☐

Remark 2.2. In the above proof, only the last step used simply connectedness of M. More generally, we have the following. The proof is both extremely routine and awfully long, so of course it won't be included here.

Theorem 2.3. Let X be a connected, locally path connected and semi-locally simply connected space, $x \in X$. Then

(i) The category of locally constant sheaves of sets on X is equivalent to the category of left $\pi_1(X, x)$-sets.

(ii) Let R be a commutative ring. The category of locally constant sheaves of R-modules on X is equivalent to the category of left $R[\pi_1(X, x)]$-modules.

(iii) The above equivalences are given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$ and a reconstruction functor, obtained by constructing étale spaces as the quotients of the universal cover by the stabilizer of each orbit of the $\pi_1(X, x)$-action, and then taking the disjoint union of them.

The End

Fun fact: I don't know what happens if we consider smooth functions instead.

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