# The series expansion of log(2) and Abel's Theorem

math center

In this note, we prove the following formula

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{1}$$

from the series expansion of $\log(1+x)$, which is

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad |x| < 1. \tag{2}$$

Here everything in (2) are assumed to be known. We will show that (i) the series in (1) converges, (ii) given the convergence, the series in (2) is continuous near $x = 1$.

**Proposition 0.1** (Dirichlet's test)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers and denote its partial sums as $A_n := \sum_{k=1}^{n} a_k$. Suppose that the partial sum is bounded:

$$\exists M > 0,\ |A_n| \leq M \text{ for all } n \geq 1.$$

Let $\{b_n\}_{n\geq 1}$ be a sequence of real numbers that monotonically decreases to 0. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges, and we have

$$\left| \sum_{k=1}^{\infty} a_k b_k \right| \leq M b_1.$$

*Proof.* First, we derive the usual summation by parts.

$$A_{k+1} b_{k+1} - A_k b_k = a_{k+1} b_{k+1} + A_k (b_{k+1} - b_k).$$

Summing this from $k = 1$ to $n - 1$, we obtain

$$A_n b_n - A_1 b_1 = \sum_{k=2}^{n} a_k b_k + \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k).$$

The result follows easily from this. □

**Remark 0.2.** Replace $A_k$ by $a_k$ and $a_k$ by $(\Delta a)_{k-1}$ to obtain the 'more standard' summation by parts formula.

**Corollary 0.3.** Suppose that $\mu \in \mathbb{C} \setminus \{1\}$ and $|\mu| \leq 1$. Let $\{b_n\}_{n\geq 1}$ be a sequence of real numbers that monotonically decreases to 0. Then the series $\sum_{k=1}^{\infty} \mu^k b_k$ converges, and we have

$$\left| \sum_{k=1}^{\infty} \mu^k b_k \right| \leq \frac{2}{|1-\mu|} b_1.$$

*Proof.* Obvious. □

**Corollary 0.4** (Alternating series test)**.** Let $\{b_n\}_{n\geq 1}$ be a sequence of real numbers that monotonically decreases to 0. Then the series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ converges, and we have

$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}b_k\right| \leq b_1.$$

*Proof.* Obvious. □

In particular, (1) converges. Now we prove the second part.

**Theorem 0.5** (Abel)**.** Let $f(x) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series with finite radius of convergence $R > 0$. Suppose that there is a point $\zeta = z_0 + Re^{i\theta}$ such that the series $\sum_{n=1}^{\infty} a_n(\zeta - z_0)^n$ converges.

Then the power series converges uniformly on the segment $[z_0, \zeta]$. In particular, we have

$$f(\zeta) = \lim_{r\to R^-} f(z_0 + re^{i\theta}).$$

*Proof.* WLOG assume $z_0 = 0$. By considering the series $\sum_{n=1}^{\infty} a_n \left(Re^{i\theta}\right)^n z^n$ for $r \in [0, R)$, we may also assume $\zeta = 1$. Now we have $f(x) = \sum_{n=1}^{\infty} a_n z^n$ and the convergence of the series $\sum_{n=1}^{\infty} a_n$. We need to prove the result for the interval $[0, 1]$.

Fix $\varepsilon > 0$. By Cauchy's criterion, there exists $N$ such that

$$m \geq n \geq N \Longrightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon.$$

For any $n \geq N$ and $x \in [0, 1)$, Dirichlet's test applies to the sequences $\{a_{k+n}\}_{k\geq 1}$ and $\{x^{k+n}\}_{k\geq 1}$, showing that the series $\sum_{k=1}^{\infty} a_{k+n}x^{k+n} = \sum_{k=n+1}^{\infty} a_k x^k$ converges on $[0, 1)$. More importantly, we have the estimate

$$\left|\sum_{k=n+1}^{\infty} a_k x^k\right| \leq \varepsilon x^{n+1} < \varepsilon.$$

Combining with the obvious inequality $\left|\sum_{k=n+1}^{\infty} a_k\right| \leq \varepsilon$, we see that

$$n \geq N \Longrightarrow \sup_{x\in[0,1]} \left|f(x) - \sum_{k=1}^{n} a_k x^k\right| \leq \varepsilon.$$

This shows the uniform convergence on $[0, 1]$. Therefore, $f$ is continuous on $[0, 1]$. □

This proves (1).

By Corollary 0.3, it is easy to see that the series in (2) converges everywhere on the boundary $|x| = 1$ except at $x = -1$ — not just at $x = 1$. Hence with Abel's theorem we might obtain some other things. We won't do that here.

# The End

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